

# Well-Posed Boundary Condition for Acoustic Liners in Straight Ducts with Flow

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The Myers boundary condition for acoustics within flow over an acoustic lining has been shown to be ill-posed, leading to numerical stability issues in the time domain and mathematical problems with stability analyses. This paper gives a modification (for flat or cylindrical straight ducts) to make the Myers boundary condition well posed, and indeed more accurate, by accounting for a thin inviscid boundary layer over the lining and correctly deriving the boundary condition to first order in the boundary-layer thickness. The modification involves two integral terms over the boundary layer. The first may be written in terms of the mass, momentum, and kinetic-energy thicknesses of the boundary layer, which are shown to physically correspond to a modified boundary mass, modified grazing velocity, and a tension along the boundary. The second integral term is related to the critical layer within the boundary layer. A time domain version of the new boundary condition is proposed, although not implemented. The modified boundary condition is validated against high-fidelity numerical solutions of the Pridmore-Brown equation for sheared inviscid flow in a cylinder. Absolute instability boundaries are given for certain examples, though convective instabilities appear to always be present at certain frequencies for any boundary-layer thickness.

## Nomenclature

$A_{0,1,2}$	= dimensionless constants multiplying terms in the asymptotic expansion of $\tilde{p}$ within the boundary layer	$R(r)$	= mean-flow density (nondimensionalized by $R_0$ )
$a$	= duct radius, m	$R_0$	= mean-flow centerline density, $\text{kg} \cdot \text{m}^{-3}$
$B_{0,1,2}$	= dimensionless constants multiplying terms in the asymptotic expansion of $\tilde{p}$ within the boundary layer	$r$	= radial coordinate (nondimensionalized by $a$ )
$C_0$	= mean-flow centerline sound speed, $\text{m s}^{-1}$	$r^*$	= radial position of a critical layer, for which $\omega - U(r^*)k = 0$ (nondimensionalized by $a$ )
$D$	= boundary damping constant (nondimensionalized by $\gamma P_0/C_0$ )	$T$	= boundary tension constant (nondimensionalized by $\gamma P_0 a$ )
$d$	= boundary mass-per-unit area (nondimensionalized by $a R_0$ )	$t$	= time (nondimensionalized by $a/C_0$ )
$E$	= arbitrary nonzero dimensionless constant	$U$	= mean-flow fluid velocity (nondimensionalized by $C_0$ )
$e_r$	= unit vector in the radial direction	$U(r)$	= mean-flow axial velocity (nondimensionalized by $C_0$ )
$e_x$	= unit vector in the axial direction	$u$	= perturbation fluid velocity (nondimensionalized by $C_0$ )
$e_\theta$	= unit vector in the azimuthal direction	$\tilde{u}(r)$	= axial harmonic perturbation velocity (nondimensionalized by $C_0$ )
$i$	= imaginary unit (so that $i^2 = -1$ )	$v$	= fluid velocity (nondimensionalized by $C_0$ )
$J_m(x)$	= Bessel's function of the first kind of order $m$	$\tilde{v}(r)$	= radial harmonic perturbation velocity (nondimensionalized by $C_0$ )
$K$	= boundary spring constant (nondimensionalized by $\gamma P_0/a$ )	$\tilde{v}_o(r)$	= radial harmonic perturbation velocity outside the boundary layer (nondimensionalized by $C_0$ )
$k$	= axial wave number (nondimensionalized by $1/(2\pi a)$ )	$\tilde{w}(r)$	= azimuthal harmonic perturbation velocity (nondimensionalized by $C_0$ )
$L$	= depth of Helmholtz resonator cavities (nondimensionalized by $a$ )	$x$	= axial coordinate along the cylinder (nondimensionalized by $a$ )
$M$	= $U(0)$ = centerline mean-flow Mach number	$y$	= inner scaling within the boundary layer, $r = 1 - \delta y$ (nondimensionalized by $\delta a$ )
$m$	= azimuthal order (integer)	$Z$	= boundary impedance (nondimensionalized by $R_0 C_0$ )
$n$	= surface normal pointing out of the fluid	$Z_{\text{mod}}$	= boundary impedance modified to account for the $\delta I_0$ effect of the boundary layer (nondimensionalized by $R_0 C_0$ )
$\mathcal{O}(x)$	= some function $f(x)$ such that, for some $M$ and $x_0$ , $ f(x)  < M x $ for $ x  < x_0$	$\alpha$	= radial wave number (nondimensionalized by $1/a$ )
$P$	= mean-flow pressure (nondimensionalized by $\gamma P_0$ )	$\gamma$	= ratio of specific heats, $c_p/c_v$
$P_0$	= mean-flow pressure, Pa	$\delta$	= boundary-layer thickness length scale (nondimensionalized by $a$ )
$p$	= pressure (nondimensionalized by $\gamma P_0$ )	$\delta_{\text{ke}}$	= kinetic energy thickness of the boundary layer (nondimensionalized by $a$ )
$\tilde{p}(r)$	= harmonic pressure perturbation (nondimensionalized by $\gamma P_0$ )	$\delta_{\text{mass}}$	= mass thickness of the boundary layer (nondimensionalized by $a$ )
$\tilde{p}_o(r)$	= harmonic pressure perturbation outside the boundary layer (nondimensionalized by $\gamma P_0$ )	$\delta_{\text{mom}}$	= momentum thickness of the boundary layer (nondimensionalized by $a$ )

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$\eta_{\text{mod}}$	= normal boundary displacement modified to account for the $\delta I_0$ effect of the boundary layer (nondimensionalized by $a$ )
$\theta$	= azimuthal coordinate, radians
$\tilde{\theta}$	= arbitrary dimensionless number used in Sec. VI
$\pi$	= ratio of the circumference of a circle to its diameter
$\rho$	= density (nondimensionalized by $R_0$ )
$\tilde{\rho}(r)$	= harmonic density perturbation (nondimensionalized by $R_0$ )
$\omega$	= frequency, with $\exp\{i\omega t\}$ sign convention (nondimensionalized by $C_0/(2\pi a)$ )
$\cdot_y$	= $d \cdot / dy$
$\cdot_r$	= $d \cdot / dr$

## I. Introduction

MODERN turbofan aeroengines almost universally use acoustic linings within the intake and bypass ducts to reduce noise. Simple mathematical models in simple geometries and more complicated numerical simulations of more realistic geometries are used to optimize the type and placement of such linings in order to obtain the greatest benefit. Commonly, the interaction of acoustics with an acoustic lining is specified using the impedance of the lining. Taking the fluctuating pressure within the fluid to be  $p(x) \exp\{i\omega t\}$  and the velocity to be  $\mathbf{v}(x) \exp\{i\omega t\}$ , the impedance of the lining is  $Z = p/(\mathbf{v} \cdot \mathbf{n})$ , where  $\mathbf{n}$  is the surface normal pointing out of the fluid ( $Z$  will usually be a function of frequency  $\omega$ ). If the acoustics is on top of a mean flow that slips across the lining, it is well known that this boundary condition must be modified [1–7]. If the fluid velocity at the boundary is  $\mathbf{U} + \mathbf{u} \exp\{i\omega t\}$ , then the boundary condition becomes

$$i\omega \mathbf{u} \cdot \mathbf{n} = (i\omega + \mathbf{U} \cdot \nabla - (\mathbf{n} \cdot \nabla \mathbf{U}) \cdot \mathbf{n}) p / Z \quad (1)$$

This follows from matching fluid and solid normal displacement, rather than normal velocity, and is known as the Myers [7] or Ingard [4]–Myers [7] boundary condition, although this equation (apart from the final term) was earlier given by, e.g., Miles [1], Eq. 3.3'. This, in straight ducts for which the final term is identically zero, was shown to be the correct asymptotic limit of a vanishingly thin inviscid boundary layer independently by Eversman and Beckemeyer [5] and Tester [6], though the boundary needed to be extremely thin to attain this limit in some cases [6,8]. For numerical simulations, this condition therefore provides an attractive alternative to resolving the thin boundary layer over the lining.

The final term in Eq. (1) is nonzero for complicated geometries and flows for which the flow just outside the boundary layer is nonparallel (e.g., [9,10]). The term is necessary for a compliant oscillating boundary, as the boundary motion causes the boundary to see a different normal mean-flow velocity at different parts of its oscillation. However, the validity of this assumption for a fixed permeable acoustic liner is questionable (see, e.g., [11]), with the proof that Eq. (1) is the correct boundary condition for a vanishingly thin inviscid boundary layer only having been given for straight ducts [5,6]. As in those papers, in this paper we will only consider straight ducts, for which this Myers nonparallel flow term is identically zero.

Unfortunately, Eq. (1) applied with slipping flow leads to numerical instabilities (see, e.g., [9,12]) and mathematical ill-posedness [13]. To circumvent slipping flow, there has been recent interest in acoustics within sheared mean-flow [14,15], following the foundations set by Pridmore-Brown [16] and Mungur and Plumblee [17]. This paper will concentrate on the effect of a thin but finite thickness inviscid boundary layer over the acoustic lining by incorporating the first- and second-order corrections in the boundary-layer thickness. Such a situation has been investigated before [6,8,18–27], at least to first order; in particular, the derivation in the first part of Sec. III is similar to the derivation of [24], as detailed in Sec. III, although [24] do not allow for temperature or density variations within their boundary layer, nor cover the material presented in the remainder of this paper. To the authors knowledge, the present paper is the first time that a simple asymptotically valid

Myers-type boundary condition incorporating a finite thickness compressible boundary layer has been proposed, compared with high-fidelity numerical simulations of the Pridmore-Brown equation, and had a stability analysis performed.

Recently, and concurrently with the current work, Rienstra and Darau [28,29] proposed a modified Myers boundary condition based on a two-dimensional incompressible model. Their results are compared with those derived here in Sec. VI.

## II. Governing Equations

Our governing equations are the equations of motion for an inviscid compressible perfect gas [30]:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad \rho \frac{D\mathbf{u}}{Dt} = -\nabla p, \quad \frac{Dp}{Dt} = \frac{\gamma p}{\rho} \frac{D\rho}{Dt} \quad (2)$$

where  $D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ ,  $\rho$  is the density,  $\mathbf{u}$  is the velocity,  $p$  is the pressure, and  $\gamma$  is the ratio of specific heats.

Although what follows is equally valid for a rectangular duct, we will concern ourselves here with a cylindrical duct, described by  $(x, r, \theta)$  coordinates with the  $x$ -axis running along the centerline of the cylinder. We take the flow to be (for real integer  $m$ )

$$\mathbf{u} = U(r)\mathbf{e}_x + (\tilde{u}(r)\mathbf{e}_x + \tilde{v}(r)\mathbf{e}_r + \tilde{w}(r)\mathbf{e}_\theta) \exp\{i\omega t - ikx - im\theta\} \quad (3a)$$

$$p = P + \tilde{p}(r) \exp\{i\omega t - ikx - im\theta\} \quad (3b)$$

$$\rho = R(r) + \tilde{\rho}(r) \exp\{i\omega t - ikx - im\theta\} \quad (3c)$$

where the tilde denotes acoustic quantities which are assumed to be small compared with the steady mean-flow. We nondimensionalize distance by the radius of the cylinder, velocity by the centerline mean-flow speed of sound  $C_0 = \sqrt{\gamma P/R(0)}$ , and density by the centerline mean-flow density  $R(0)$ , implying that pressure is nondimensionalized by  $\gamma P$  where  $P$  is the mean-flow pressure. Under this nondimensionalization, the duct wall is at  $r = 1$ ,  $R(0) = 1$ ,  $P = 1/\gamma$ ,  $M = U(0)$  is the centerline mean-flow Mach number, and  $\omega$  is the Helmholtz number.

Note that  $U(r)$  and  $R(r)$  may be arbitrary functions of  $r$  as far as this paper is concerned. In reality, gradients of  $U(r)$  are caused by viscosity and gradients of  $R(r)$  (since  $R \propto P/T$ , where  $T$  is the temperature) are caused by thermal conduction and viscous dissipation (see, e.g., [11] for further details). These mechanisms are here considered insignificant for the acoustic perturbations, which is the standard acoustic assumption, and are considered to have been already accounted for in the mean-flow through nonconstant  $U(r)$  and  $R(r)$ .

Substituting Eqs. (3a–3c) into Eq. (2) and taking only terms linear in the acoustic quantities gives the linearized mass, momentum and energy equations as

$$\begin{aligned} i(\omega - Uk)\tilde{\rho} &= -\tilde{v} \frac{dR}{dr} - \frac{R}{r} \frac{d(r\tilde{v})}{dr} + \frac{i}{r} m R \tilde{w} + ik R \tilde{u} \\ iR(\omega - Uk)\tilde{v} &= -\frac{d\tilde{p}}{dr}, \quad iR(\omega - Uk)\tilde{w} = \frac{im}{r} \tilde{p} \\ iR(\omega - Uk)\tilde{u} &= ik\tilde{p} - R \frac{dU}{dr} \tilde{v} \\ i(\omega - Uk)\tilde{p} &= \frac{1}{R} \left( i(\omega - Uk)\tilde{\rho} + \frac{dR}{dr} \tilde{v} \right) \end{aligned}$$

which, after eliminating every acoustic variable but  $\tilde{p}$ , gives the Pridmore-Brown [16] equation in cylindrical form:

$$\tilde{p}'' + \left( \frac{1}{r} + \frac{2kU'}{\omega - Uk} - \frac{R'}{R} \right) \tilde{p}' + \left( R(\omega - Uk)^2 - k^2 - \frac{m^2}{r^2} \right) \tilde{p} = 0 \quad (4)$$

where a prime denotes  $d/dr$ . The radial velocity  $\tilde{v}$ , needed for the boundary condition, is given by  $\tilde{v} = i\tilde{p}'/(R(\omega - Uk))$ .

The boundary condition to be applied at  $r = 0$  is to require  $\tilde{p}$  to be regular. The boundary condition to be applied at  $r = 1$  is  $\tilde{p}/\tilde{v} = Z$ , giving

$$i\omega Z\tilde{p}' - \omega^2 R(1)\tilde{p} = 0$$

The Myers boundary condition [Eq. (1)], under the assumption that the mean flow is constant apart from an infinitely thin boundary layer at  $r = 1$ , predicts that

$$i\omega Z\tilde{p}' - (\omega - Mk)^2\tilde{p} = 0 \quad (5)$$

It is this boundary condition that we propose to modify here.

### III. Derivation of the Modified Boundary Condition

In this section, we consider a thin boundary layer about  $r = 1$  of typical width  $\delta$ , outside which the mean-flow is uniform, so that  $U(r) = M$  and  $R(r) = 1$  for the majority of the flow. We will consider the limit of small  $\delta$  with all other parameters remaining  $\mathcal{O}(1)$  independent of  $\delta$ . The asymptotic procedure used here is similar to that of [24], except that [24] worked with an expansion of  $\log(\tilde{p})$ , while here we will work directly with the physical variables  $\tilde{p}$  and  $\tilde{v}$ , and that here we allow for nonconstant density  $R(r)$  (and hence nonconstant temperature) within the boundary layer.

If the boundary layer did not exist, the solution to Eq. (4) regular at  $r = 0$  would be

$$\tilde{p}_o(r) = EJ_m(\alpha r) \quad \text{where } \alpha^2 = (\omega - Mk)^2 - k^2 \quad (6)$$

Accounting for the boundary layer to first order in  $\delta$ , the asymptotics (the details of which are given in the Appendix) give a composite asymptotic solution of

$$\begin{aligned} \tilde{p}(r) = & EJ_m(\alpha r) - \alpha EJ'_m(\alpha) \int_0^r 1 - \frac{(\omega - U(r)k)^2 R(r)}{(\omega - Mk)^2} dr \\ & + \mathcal{O}(\delta^2) \end{aligned} \quad (7)$$

This shows very good agreement with the numerics, as is shown in section Sec. IV. However, in order to derive the correction to the Myers boundary condition to  $\mathcal{O}(\delta)$ , it turns out to be necessary to derive the pressure within the boundary layer to  $\mathcal{O}(\delta^2)$ . This is again done in the Appendix, with the result that, evaluated at the wall lining  $r = 1$ ,

$$\begin{aligned} \tilde{p}(1) = & EJ_m(\alpha) - \alpha EJ'_m(\alpha)\delta I_0 + \mathcal{O}(\delta^2) \\ \tilde{v}(1) = & \frac{i(\omega - U(1)k)}{(\omega - Mk)^2} [\alpha EJ'_m(\alpha) - (k^2 + m^2)\delta I_1 EJ_m(\alpha) + \mathcal{O}(\delta^2)] \end{aligned}$$

with

$$\begin{aligned} \delta I_0 = & \int_0^1 1 - \frac{(\omega - U(r)k)^2 R(r)}{(\omega - Mk)^2} dr \\ \delta I_1 = & \int_0^1 1 - \frac{(\omega - Mk)^2}{(\omega - U(r)k)^2 R(r)} dr \end{aligned} \quad (8)$$

Note that, since the integrands of  $\delta I_0$  and  $\delta I_1$  are identically zero outside the boundary layer,  $\delta I_0$  and  $\delta I_1$  are indeed both of order  $\delta$ .

By stipulating that  $Z = \tilde{p}(1)/\tilde{v}(1)$ , we arrive at the dispersion relation

$$\begin{aligned} i(\omega - U(1)k)Z[\alpha J'_m(\alpha) - (k^2 + m^2)\delta I_1 J_m(\alpha)] \\ - (\omega - Mk)^2[J_m(\alpha) - \alpha J'_m(\alpha)\delta I_0] = 0 \end{aligned}$$

Our generalization of the Myers boundary condition is finally arrived at by substituting  $\tilde{p}_o = EJ_m(\alpha)$  and  $\tilde{v}_o = i\alpha EJ'_m(\alpha)/(\omega - Mk)$ , meaning that  $\tilde{p}_o$  and  $\tilde{v}_o$  are the pressure and normal velocity at the wall obtained by ignoring the boundary layer. This gives the boundary condition

$$Z\left[\tilde{v}_o + \frac{k^2 + m^2}{i(\omega - Mk)}\delta I_1\tilde{p}_o\right] = \frac{\omega - Mk}{\omega - U(1)k}[\tilde{p}_o + i(\omega - Mk)\delta I_0\tilde{v}_o] \quad (9)$$

This is the main result of the paper. For most purposes, no slip implies that  $U(1) = 0$ , although setting  $U(1) \neq 0$  in order to model surface roughness (as in [26]) is also possible.

#### A. Interpretation of the $\delta I_0$ Term

At least part of the modified boundary condition [Eq. (9)] may be interpreted physically. Let us first define the boundary-layer mass, momentum, and kinetic energy thicknesses of the boundary layer to be

$$\begin{aligned} \delta_{\text{mass}} = & \int_0^1 1 - R(r) dr, \quad \delta_{\text{mom}} = \int_0^1 1 - \frac{R(r)U(r)}{M} dr \\ \delta_{\text{ke}} = & \int_0^1 1 - \frac{R(r)U(r)^2}{M^2} dr \end{aligned}$$

Multiplying out  $\delta I_0$  gives

$$\delta I_0 = \frac{1}{(\omega - Mk)^2} (\omega^2 \delta_{\text{mass}} - 2\omega k M \delta_{\text{mom}} + k^2 M^2 \delta_{\text{ke}})$$

which, when substituted into Eq. (9) gives the boundary condition

$$\begin{aligned} \tilde{v}_o[i(\omega - U(1)k)Z + \omega^2 \delta_{\text{mass}} - 2\omega k M \delta_{\text{mom}} + k^2 M^2 \delta_{\text{ke}}] \\ = \tilde{p}_o \left[ i(\omega - Mk) - \frac{\omega - U(1)k}{\omega - Mk} Z(k^2 + m^2)\delta I_1 \right] \end{aligned}$$

The  $\delta I_0$  term may therefore be thought of as modifying the impedance seen by the acoustics from the actual impedance of the boundary,  $Z$ , to an effective impedance  $Z_{\text{mod}}$  given by

$$\begin{aligned} i(\omega - U(1)k)Z_{\text{mod}} = & i(\omega - U(1)k)Z + \omega^2 \delta_{\text{mass}} \\ & - 2\omega k M \delta_{\text{mom}} + k^2 M^2 \delta_{\text{ke}} \end{aligned}$$

These terms have a physical interpretation. To see this, first consider a theoretical acoustic lining modeled as a flexible impermeable sheet, whose displacement  $\eta$  in the normal direction pointing out of the fluid is governed by

$$d \frac{\partial^2 \eta}{\partial t^2} = \tilde{p} - K\eta - D \frac{\partial \eta}{\partial t} + T \frac{\partial^2 \eta}{\partial x^2} \quad (10)$$

Here,  $\tilde{p}$  is a forcing term due to the acoustic pressure,  $d$  is a mass density,  $K$  is a spring constant,  $D$  is a damping constant, and  $T$  is an elastic tension in the sheet. Assuming  $\eta$  and  $\tilde{p}$  to have  $\exp\{i\omega t - ikx\}$  dependence gives

$$i(\omega - U(1)k)Z = \frac{\tilde{p}}{\eta} = -\omega^2 d + i\omega D + K + k^2 T$$

The modified impedance  $Z_{\text{mod}}$  based on this theoretical boundary model is therefore

$$\begin{aligned} i(\omega - U(1)k)Z_{\text{mod}} = & \frac{\tilde{p}}{\eta_{\text{mod}}} = -\omega^2(d - \delta_{\text{mass}}) + i\omega D \\ & + K + k^2(T + M^2 \delta_{\text{ke}}) - 2\omega k M \delta_{\text{mom}} \end{aligned} \quad (11)$$

which could be interpreted as a theoretical boundary with displacement  $\eta_{\text{mod}}$  and governing equation

$$\begin{aligned} (d - \delta_{\text{mass}}) \left( \frac{\partial}{\partial t} - \frac{M \delta_{\text{mom}}}{d - \delta_{\text{mass}}} \frac{\partial}{\partial x} \right)^2 \eta_{\text{mod}} = & \tilde{p} - K\eta_{\text{mod}} - D \frac{\partial \eta_{\text{mod}}}{\partial t} \\ & + \left( T + M^2 \left( \delta_{\text{ke}} + \frac{\delta_{\text{mom}}^2}{d - \delta_{\text{mass}}} \right) \right) \frac{\partial^2 \eta_{\text{mod}}}{\partial x^2} \end{aligned} \quad (12)$$

The modified boundary-layer term  $\delta I_0$  therefore represents three physical effects:

- 1) The mass deficit  $\delta_{\text{mass}}$  in the boundary layer causes the effective boundary seen by the acoustics to be lighter.
- 2) The momentum deficit  $M\delta_{\text{mom}}$  in the boundary layer causes the advection (accounted for in the Myers boundary condition at a velocity  $M$ ) to be corrected by the effective velocity deficit of the boundary layer,  $M\delta_{\text{mom}}/(d - \delta_{\text{mass}})$ .
- 3) The kinetic energy deficit  $\frac{1}{2}M^2\delta_{\text{ke}}$  in the boundary layer, together with the momentum deficit, instantiate themselves as a tension along the boundary.

Note from Eq. (11) that these effects become significant when  $k$  and  $\omega$  are  $\mathcal{O}(\delta^{-1/2})$ , and that in particular the effective tension becomes significant irrespective of  $\omega$  when  $k = \mathcal{O}(\delta^{-1/2})$ ; these values of  $k$  and  $\omega$  remain within the range of validity of the asymptotic derivation given in the Appendix. The effect of tension along the boundary is particularly interesting, since it suggests there may be unforced travelling waves permitted along the boundary (which would be damped if  $D \neq 0$ ). Note that if  $T = 0$  the uncorrected boundary displacement [Eq. (10)] is local and does not support or prevent travelling waves, whereas the modified boundary [Eq. (12)] is always nonlocal and does support travelling waves (this effect was previously noted by Eversman [25]). In the limit  $\delta \rightarrow 0$ , this added tension disappears and, at fixed frequency, the wave numbers for these travelling waves tend to infinity as  $\delta \rightarrow 0$ . This may well correspond to the ill-posedness and instability of the Myers boundary condition at arbitrarily short wavelengths [13], although this will not be pursued further here.

### B. Interpretation of the $\delta I_1$ Term

The  $\delta I_1$  term is important, as it appears to be this term that is responsible for the well-posedness of the modified boundary condition. However, the interpretation of the  $\delta I_1$  term is tricky, and no physical explanation of it is given here. It is worth noting that the presence a critical layer at  $r = r^*$  (where  $r^*$  is possibly complex), for which  $\omega - U(r^*)k = 0$ , may have a significant effect on  $\delta I_1$ , especially if  $|(1 - r^*)/\delta| \ll 1$ . The case  $r^* \in [0, 1]$  is explicitly excluded from our analysis, as in this case  $\delta I_1$  may not be well defined; this is in line with previous analyses (see, e.g., [24]). For fixed  $\omega$ , this exclusion causes a branch cut in the  $k$ -plane. This branch cut may be removable, as in the case of a constant-then-linear velocity profile given next, in which case  $\delta I_1$  may be defined for  $r^* \in [0, 1]$  by continuity.

Asymptotic values of  $\delta I_1$  may be calculated, and it can be shown that, for  $U(1) = 0$ ,

$$\delta I_1 \sim \int_0^1 1 - 1/R(r) dr + \frac{2kM}{\omega} \int_0^1 1 - \frac{U(r)}{MR(r)} dr \quad \text{for } \omega/k \gg 1$$

$$\delta I_1 \sim \frac{-M^2k}{\omega R(1)U'(1)} \quad \text{for } \omega/k \ll 1$$

Moreover, for a constant density  $R(r) \equiv 1$  and a constant-then-linear boundary-layer profile,

$$U(r) = \begin{cases} M(1-r)/\delta & (1-r) < \delta \\ M & (1-r) > \delta \end{cases} \quad (13)$$

$\delta I_1$  may be directly calculated to give  $\delta I_1 = \delta M k / \omega$ .

### C. Modified Boundary Condition in the Time Domain

To apply the new modified boundary condition in the time domain, Eq. (9) needs some coercing into a suitable form. One possible way to do this would be to assume that the density is uniform and the boundary-layer profile is constant-then-linear [Eq. (13)] with a given momentum thickness  $\delta_{\text{mom}} = \delta/2$ . Then Eq. (9) may be rearranged to give

$$\frac{\partial}{\partial t} \left( \frac{\partial}{\partial t} + M \frac{\partial}{\partial x} \right) \tilde{v} + 2M\delta_{\text{mom}} \frac{\partial}{\partial x} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \theta^2} \right) \tilde{p}$$

$$= \left( \frac{\partial}{\partial t} + M \frac{\partial}{\partial x} \right)^2 \tilde{v}_b$$

where  $\tilde{v}_b = \tilde{p}/Z_{\text{mod}}$  is the velocity of the boundary using a modified boundary model incorporating the  $\delta I_0$  terms from Sec. III.A. For example, if the desired boundary model in the time domain were as given in Eq. (10), then  $\tilde{v}_b$  would be given by  $\tilde{v}_b = \partial \eta_{\text{mod}} / \partial t$  with  $\eta_{\text{mod}}$  satisfying Eq. (12).

## IV. Comparison with Numerics

To validate the preceding asymptotics, the asymptotic solution will be compared with numerical solutions to the full Pridmore-Brown Eq. (4). The numerical solutions were generated using an implicit 12th-order symmetric finite difference discretization with unevenly spaced collocation points, so that the collocation points could be clustered within the boundary layer. Typically, 8000 points were used, with at least 400 points within the boundary layer irrespective of the width of the boundary layer. The  $r = 1$  boundary condition used was  $\tilde{p}(1) = 1$ ; a Newton-Raphson iteration was then used to find roots of the dispersion relation  $\tilde{p}/\tilde{v} = Z$ .

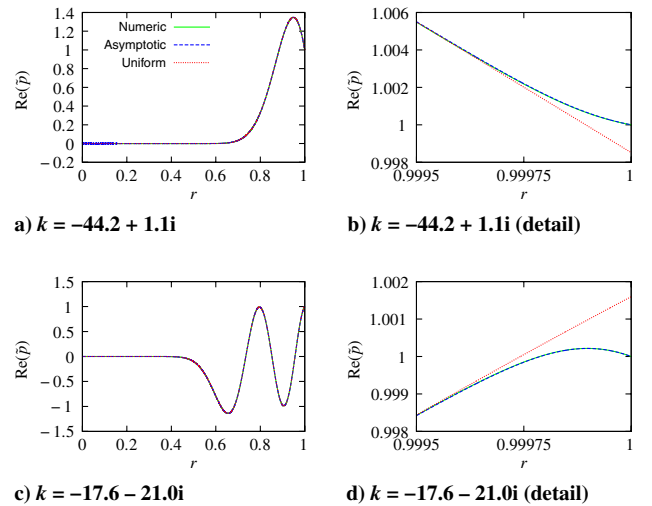
The boundary-layer profile used for most examples given here was chosen to be the tanh profile, as used by Rienstra and Vilenski [27]:

$$U(r)/M = \tanh\left(\frac{1-r}{\delta}\right)$$

$$+ (1 - \tanh(1/\delta)) \left( \frac{1 + \tanh(1/\delta)}{\delta} r + (1+r) \right) (1-r) \quad (14)$$

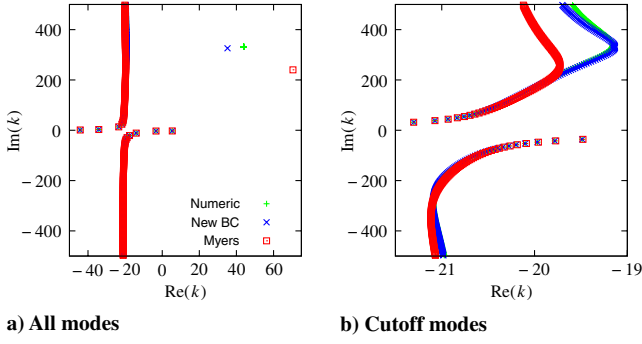
with constant density  $R(r) \equiv 1$ .

For this profile with  $\delta = 2 \times 10^{-4}$ , two comparisons between the numerical solution to Eq. (4), the first-order-accurate composite solution [Eq. (7)] and the uniform solution are given in Fig. 1. Figure 2 gives a comparison between duct modes satisfying the impedance boundary condition using the numerics, the Myers boundary condition, and the modified boundary condition [Eq. (9)]. Despite this being for such a thin boundary ( $\delta = 2 \times 10^{-4}$ ), the highly cutoff modes and the surface mode in the upper-right quadrant are noticeably affected by the finite thickness of the boundary layer. The general trend that surface modes, more highly-attenuated modes and upstream-propagating modes are more affected by the boundary layer could be due to the nature of these modes, which oscillate significantly at the boundary, while modes which oscillate less on the boundary (such as the downstream-propagating cuton modes) see less of an effect.



**Fig. 1** Plots of  $\text{Re}(\tilde{p}(r))$  against  $r$ , comparing the numerical solution to the Pridmore-Brown Eq. (4), the first-order asymptotic solution [Eq. (7)], and the uniform-flow solution [Eq. (6)].  $\omega = 31$ ,  $m = 24$ , and  $U(r)$  is given by Eq. (14) with  $M = 0.5$  and  $\delta = 2 \times 10^{-4}$ . The values of  $k$  used correspond to a-b) cuton upstream-propagating mode and c-d) partially cutoff downstream-propagating mode, if the lining has impedance  $Z = 2 + 0.6i$ .



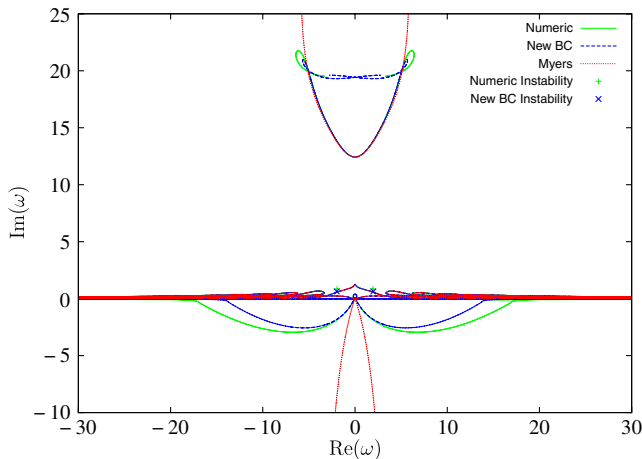


**a) All modes** **b) Cutoff modes**  
**Fig. 2** Comparison of axial wave numbers  $k$ , plotted in the  $k$ -plane, for fully numerical solutions and solutions of the modified [Eq. (9)] and original [Eq. (5)] Myers boundary conditions.  $\omega = 31$ ,  $m = 24$ ,  $Z = 2 + 0.6i$ , and  $U(r)$  is given by Eq. (14) with  $M = 0.5$  and  $\delta = 2 \times 10^{-4}$ .

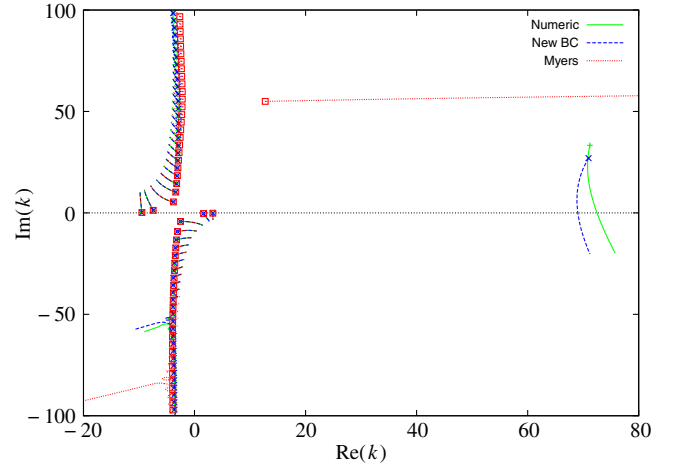
## V. Stability and Well-Posedness

One major need for a generalization of the Myers boundary condition is that the Myers boundary condition is ill posed [13], manifesting itself as numerical instability [9,12], which we hope to alleviate using this modified boundary condition [Eq. (9)]. Whatever the boundary condition at  $r = 1$ , it specifies either allowable values of  $\omega$  if  $k$  is given, or allowable values of  $k$  if  $\omega$  is given; these are here referred to as modes. For the problem to be well posed, there should be a lower bound to  $\text{Im}(\omega(k))$  for real  $k$ . For further explanation, see [13].

Figure 3 shows  $\omega(k)$  as  $k$  is varied with  $\text{Im}(k) = 0$ . The parameters used are the taken from Rienstra and Vilenski [27]. Note that for both the numerical solution and the modified boundary condition,  $\text{Im}(\omega(k))$  is bounded below (say by  $-5$ ), while the Myers solution is unbounded. Hence, the numerical and modified boundary condition solutions are well posed. Since they are well posed, we may apply the Briggs–Bers stability analysis (see Briggs [31] and Bers [32]) to these cases. As there are real values of  $k$  with  $\text{Im}(\omega(k)) < 0$ , there is a convective instability present. However, we must also look for absolute instabilities, where the system chooses its own preferred frequency and disturbances at that frequency grow exponentially in time. Absolute instability occurs for values of  $\omega$  with  $\text{Im}(\omega) < 0$  for which two modes collide in the  $k$ -plane giving a double root. Only if this collision is between a mode originating for large  $\text{Im}(\omega)$  from the lower-half  $k$ -plane and one originating from the upper-half  $k$ -plane does this then signify an absolute instability (see [13], for details). For this example, such a pinch does occur (as shown in Fig. 3), but for  $\text{Im}(\omega) > 0$ , so that in this case no absolute instabilities are present.



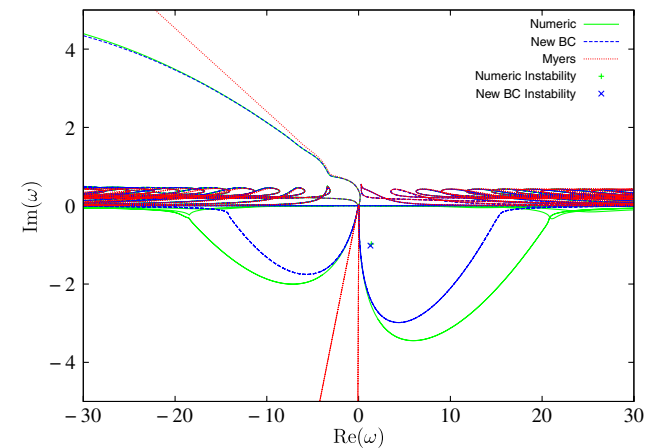
**Fig. 3** Plotted in the  $\omega$ -plane are trajectories of  $\omega(k)$  as  $k$  is varied with  $k$  real, for the numerically calculated roots, solutions to the modified boundary condition [Eq. (9)], and the original Myers boundary condition [Eq. (5)].  $\times$  and  $+$  denote values of  $\omega$  for which two  $k$ -roots coincide (a double root).  $m = 0$ ,  $Z = 3 + 0.15i\omega - 1.15i/\omega$ , with  $U(r)$  given by Eq. (14) with  $M = 0.5$  and  $\delta = 2 \times 10^{-3}$ .



**Fig. 4** Trajectories of modes in the  $k$ -plane as  $\omega$  is varied from 5 to  $5 - 5i$ . The points are for  $\omega = 5$ . Since the modes on the far right cross the real  $k$ -axis as  $\text{Im}(\omega)$  is varied, they correspond to downstream-propagating instabilities. Parameters are (as in Fig. 3)  $m = 0$ ,  $Z = 3 + 0.15i\omega - 1.15i/\omega$ , with  $U(r)$  given by Eq. (14) with  $M = 0.5$  and  $\delta = 2 \times 10^{-3}$ .

However, as the boundary-layer thickness  $\delta$  is reduced, this double root moves into the lower-half  $\omega$ -plane and produces an absolute instability. The critical value of  $\delta$  for which  $\text{Im}(\omega) = 0$  for this double root is found numerically to be  $8.6 \times 10^{-4}$  for the numerics, occurring at  $\omega = 4.3$ ,  $k = 19.8 + 95.3i$ , and  $\delta = 9.7 \times 10^{-4}$  using the new modified boundary condition, occurring at  $\omega = 3.7$  and  $k = 22.8 + 82.1i$ .

For values of  $\delta$  sufficiently large that there is no absolute instability, the system here is convectively unstable, with a downstream-propagating mode that has  $\text{Im}(k) > 0$ . This can be seen by applying the Briggs–Bers (see Briggs [31] and Bers [32]) criterion, which says that no convective instabilities are present provided  $\text{Im}(\omega)$  is sufficiently negative; i.e., below all  $\omega(k)$  for real  $k$ . We may therefore ascertain the stability of modes by tracking them in the  $k$ -plane as  $\omega$  is varied from sufficiently imaginary to real, with those that originate in the low-half  $k$ -plane propagating in the positive  $x$ -direction and those that originate in the upper-half  $k$ -plane propagating in the negative  $x$ -direction. This is done for  $\text{Re}(\omega) = 5$  and  $\text{Im}(\omega) \in [-5, 0]$  in Fig. 4. All modes but one are seen to be stable. The unstable mode on the far right of Fig. 4 is the mode previously suggested by Rienstra [33] as a possible hydrodynamic instability mode. Note that the original Myers boundary condition fails to correctly predict the behavior of



**Fig. 5** Trajectories of  $\omega(k)$  as  $k$  is varied with  $k$  real, for the numerically calculated roots, solutions to the modified boundary condition [Eq. (9)], and the original Myers boundary condition [Eq. (5)].  $\times$  and  $+$  denote values of  $\omega$  for which two  $k$ -roots coincide (a double root).  $m = 0$  and  $Z = 3 + 1.39i$ , with  $U(r)$  given by Eq. (14) with  $M = 0.5$  and  $\delta = 2 \times 10^{-3}$ .

this mode, while the modified boundary condition gives an accurate prediction.

There are several different models for the impedance  $Z$ . For single-frequency simulations a common assumption is that  $Z$  is constant. However, even in these situations any dependence of  $Z$  on  $\omega$  would be important for stability. To investigate solutions for fixed  $Z$ , the same situation as Fig. 3 but with  $Z$  fixed at a constant value of  $3 + 1.39i$  (an unfeasible impedance in reality; see [34]) is shown in Fig. 5. This demonstrates that the new boundary condition also regularizes this problem even if the impedance is taken to be constant, although in this case an absolute instability is now present, as seen in the figure.

Recently, Rienstra and Darau [28] performed a similar analysis using a two-dimensional incompressible model, a constant-then-linear velocity profile [Eq. (13)] and a Helmholtz resonator model for the impedance,  $Z = D + i\omega - i \cot(\omega L)$ . Nondimensionalizing their parameters under the assumption of a 0.2-m-diam cylindrical duct (chosen to exaggerate any effect of the cylindrical geometry), the corresponding stability graph of  $\omega(k)$  for real  $k$  is given in Fig. 6. In this case, Rienstra and Darau predict absolute instability for  $\delta < 3.6 \times 10^{-4}$ . Here, the modified boundary condition and the numerics both predict a critical value for  $\delta$  of  $3.7 \times 10^{-4}$ , which is in good agreement with Rienstra and Darau considering their model was of a flat surface and incompressible flow, and here the duct is cylindrical and the flow compressible. Further comparisons with the model of Rienstra and Darau are given in Sec. VI.

So far, all of these results have assumed an axisymmetric mode with azimuthal order  $m = 0$ . However, the new boundary condition multiplies the  $\delta I_1$  term by  $k^2 + m^2$ , and so large values of  $m$  may lead to different behavior. However, similar simulations to those shown here demonstrate that even in this case the problem is well posed.

The critical values for  $\delta$  found to date are given in Table 1. They show a good agreement between the modified boundary condition and the numerics, with very good agreement for thin boundary layers (as is to be expected from a correction that is asymptotically valid in the small- $\delta$  limit). In all these cases, the system was still convectively unstable for a boundary layer of width  $\delta = 0.1$ , though as  $\delta$  is made

progressively larger the frequency range for which convective instabilities can occur become smaller. For very small values of  $\delta$  the solution approximates the Myers solution until  $k$  becomes sufficiently large that the  $\mathcal{O}(\delta)$  term in the modified boundary condition become important and the problem remains regularized. Setting  $\delta = 0$  yields exactly the Myers boundary condition and the problem once again becomes ill posed.

## VI. Comparison with Rienstra and Darau's Boundary Condition

As mentioned previously, Rienstra and Darau [28] recently (and concurrently with the work presented here) analyzed the stability of an incompressible two-dimensional shear layer over an acoustic lining. As part of this work, they proposed a modified Myers boundary condition ([28], Eq. 25) which, written in the notation used here, is

$$Z = \frac{i(\omega - Mk)\tilde{p}_o - [2\delta_{\text{mom}}Mk\omega(1 - \omega/(Mk)) - \delta_{\text{ke}}M^2k^2/2]\tilde{v}_o}{i\omega\tilde{v}_o - 2\delta_{\text{mom}}k^2\tilde{p}_o} \quad (15)$$

Putting the modified Myers boundary condition given here [Eq. (9)] into a similar form for the same constant-then-linear boundary-layer profile [Eq. (13)] with  $R \equiv 1$  gives

$$Z = \frac{i(\omega - Mk)\tilde{p}_o + [2\delta_{\text{mom}}Mk\omega - \delta_{\text{ke}}M^2k^2]\tilde{v}_o}{i\omega\tilde{v}_o - 2\delta_{\text{mom}}(k^2 + m^2)(1 - \omega/(Mk))^{-1}\tilde{p}_o} \quad (16)$$

Comparing Eqs. (15) and (16) shows them to be very similar, although not identical, with both reducing to the unmodified Myers boundary condition in the limit  $\delta \rightarrow 0$ . More recently, Rienstra and Darau [29] pointed out that such boundary conditions are not unique, and that Eq. (15) may be written for any  $\tilde{\theta}$  as (again using the notation of this paper)

$$Z = \frac{i(\omega - Mk)\tilde{p}_o + [2\delta_{\text{mom}}(1 - \tilde{\theta})\omega^2 - 2\delta_{\text{mom}}(1 - 2\tilde{\theta})Mk\omega + \delta_{\text{ke}}(1 - 3\tilde{\theta})M^2k^2/2]\tilde{v}_o}{i\omega\tilde{v}_o - 2\delta_{\text{mom}}k^2\tilde{p}_o - 2i\omega\delta_{\text{mom}}\tilde{\theta}\tilde{v}_o'} \quad (17)$$

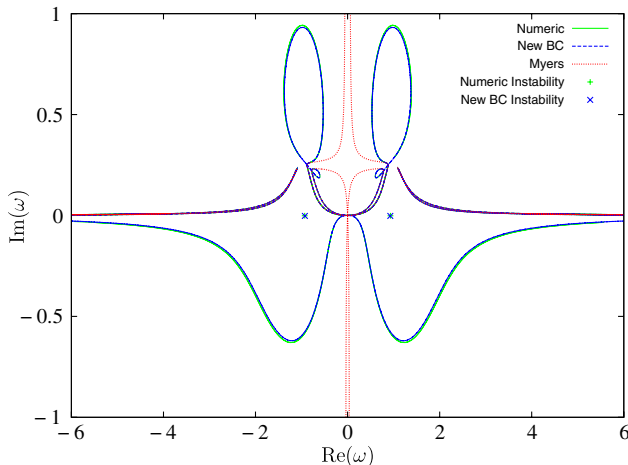


Fig. 6 Trajectories of  $\omega(k)$  as  $k$  is varied with  $k$  real for the numerically calculated roots, solutions to the modified boundary condition [Eq. (9)], and the original Myers boundary condition [Eq. (5)].  $\times$  and  $+$  denote values of  $\omega$  for which two  $k$ -roots coincide (a double root).  $m = 0$  and  $Z = 2 + 3.9i\omega - i \cot(0.3\omega)$ , with  $U(r)$  given by Eq. (13) with  $M = 0.176$  and  $\delta = 3.6 \times 10^{-4}$ .

where  $\tilde{v}_o'$  denotes the radial derivative of the outer uniform-flow solution  $\tilde{v}_o(r)$  evaluated at  $r = 1$ . This equation is Eq. 4.4 of [29]. In the two-dimensional incompressible case considered by [28,29]  $\tilde{v}_o' = ik^2\tilde{p}_o/(\omega - Mk)$ , and therefore in this case Eq. (17) and the boundary condition derived here [Eq. (16)] are identical for  $\tilde{\theta} = 1$ ; hence, the boundary conditions of [28,29] are asymptotically equivalent to Eq. (16) for two-dimensional incompressible perturbations. Rienstra and Darau [29] went on to propose  $\tilde{\theta} = 1/3$  as a giving particularly pleasing results. However, the identity  $\tilde{v}_o' = ik^2\tilde{p}_o/(\omega - Mk)$  is only valid for two-dimensional incompressible perturbations, with the equivalent here being  $\tilde{v}_o' = -\tilde{v} - i((\omega - Mk)^2 - k^2 - m^2)\tilde{p}_o/(\omega - Mk)$ ; therefore, the modified boundary conditions proposed in [28], Eq. 25, and [29], Eq. 4.4, corresponding to  $\tilde{\theta} = 0$  and  $\tilde{\theta} = 1/3$  in Eq. (17) respectively, are not asymptotically equivalent to the modified boundary condition proposed here when applied to a compressible fluid.

It should be noted that Eq. (16) is also not unique, as multiplying both numerator and denominator by  $1 - 2i\delta_{\text{mom}}(1 - \tilde{\theta})(\omega - Mk)\tilde{v}_o'/\tilde{p}_o$  and using the identity  $\tilde{v}_o'/\tilde{p}_o = (\omega - Mk)/(\omega Z) + \mathcal{O}(\delta)$  gives the asymptotically equivalent expression for arbitrary  $\tilde{\theta}$  (where  $\tilde{\theta}$  is  $\mathcal{O}(1)$ ),

**Table 1** Table of critical boundary-layer thicknesses  $\delta$ , for several different situations: for smaller values of  $\delta$ , the system is absolutely unstable; for larger values of  $\delta$ , the system is at most convectively unstable

$U(r)$ profile	$M$	$m$	$Z$	$\delta$ numerical	$\delta$ asymptotic
Tanh	0.5	0	$3 + 0.15i\omega - 1.15i/\omega$	$8.6 \times 10^{-4}$	$9.7 \times 10^{-4}$
Tanh	0.5	0	$3 + 1.39i$	$3.0 \times 10^{-2}$	$8.2 \times 10^{-2}$
Tanh	0.5	24	$2 + 0.8i\omega - i \cot(0.06\omega)$	$1.7 \times 10^{-3}$	$1.7 \times 10^{-3}$
Linear	0.176	0	$2 + 3.9i\omega - i \cot(0.3\omega)$	$3.7 \times 10^{-4}$	$3.7 \times 10^{-4}$

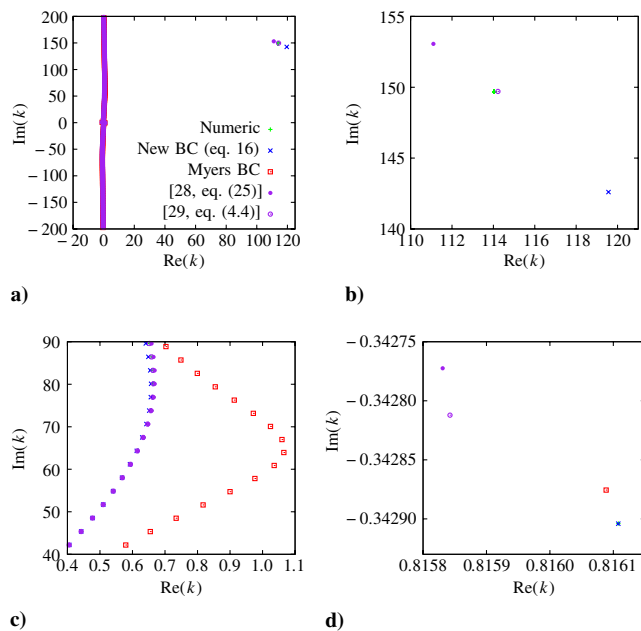
$$Z = \frac{i(\omega - Mk)\tilde{p}_o + [2\delta_{\text{mom}}(1 - \tilde{\theta})\omega^2 - 2\delta_{\text{mom}}(1 - 2\tilde{\theta})Mk\omega + \delta_{\text{ke}}(1 - 3\tilde{\theta})M^2k^2/2]\tilde{v}_o}{i\omega\tilde{v}_o - 2\delta_{\text{mom}}(k^2 + m^2)\tilde{p}_o + \frac{2\delta_{\text{mom}}}{\omega(\omega - Mk)}[\omega^2(k^2 + m^2) + (\omega - Mk)^4(1 - \tilde{\theta})/Z^2]\tilde{p}_o}$$

However, here we will continue with  $\tilde{\theta} = 1$ , for which this reduces to the originally derived Eq. (16), in order to avoid the term involving  $(\omega - Mk)^4/Z^2$  in the denominator.

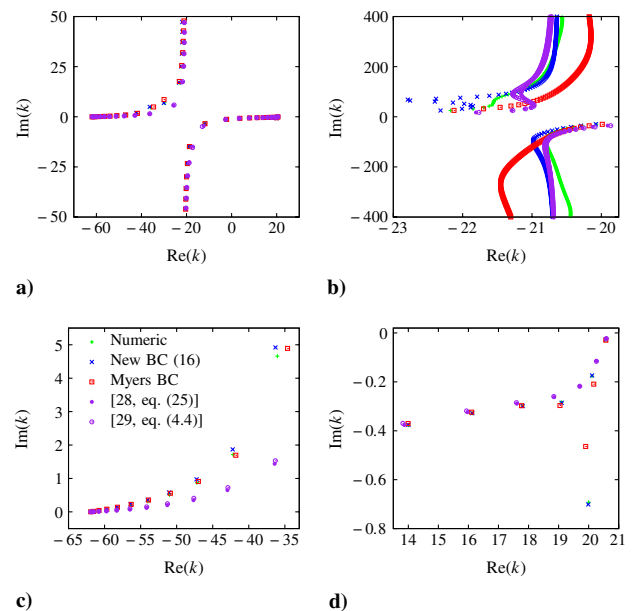
We now compare the boundary condition derived here [Eq. (16)] with the boundary condition of Rienstra and Darau [Eq. (17)] for  $\tilde{\theta} = 0$  (as originally derived in [28]) and  $\tilde{\theta} = 1/3$  (as proposed in [29]), as well as with the unmodified Myers boundary condition and the numerical solution, using a constant-then-linear boundary-layer profile. Figure 7 gives such a comparison for the parameters used by Rienstra and Darau [28], for  $\omega = 1$  and  $m = 0$ . The surface mode (shown in detail in Fig. 7b) is not correctly predicted by the unmodified Myers boundary condition, while all of the modified boundary conditions give reasonable estimates. The estimate of Rienstra and Darau for  $\tilde{\theta} = 1/3$  is the most accurate in this case; this is expected, as the surface modes are inherently an incompressible phenomena [33], and therefore the flexibility of Rienstra and Darau's boundary condition for varying  $\tilde{\theta}$  may be used without penalty. Note that their  $\tilde{\theta} = 0$  estimate is comparably as accurate as the one for the boundary condition given here. For the downstream-propagating plane wave shown in Fig. 7d, however, the situation is reversed, with

the boundary condition proposed here giving superb accuracy while both boundary conditions of Rienstra and Darau are less accurate than the unmodified Myers boundary condition. This too is expected, as it is for modes such as this that compressibility is important. Figure 7c indicates that the cutoff modes are almost equally well modeled by all of the modified boundary conditions (with, if anything, Rienstra and Darau's  $\tilde{\theta} = 1/3$  being the most accurate in this case), with the unmodified Myers boundary condition being notably less accurate.

We now move to a more aeroacoustically relevant Helmholtz number of  $\omega = 31$  and Mach number  $M = 0.5$ , with a thicker boundary layer of  $\delta = 10^{-2}$  intended to be more indicative of a fully developed turbulent boundary layer (and chosen so as to exaggerate any discrepancies between the exact and asymptotic predictions). Taking the lining to consist of Helmholtz resonators of depth  $L = 35$  mm with a facing sheet of mass reactance  $d/R(1) = 20$  mm (as used in [29]), and taking the duct to have radius 1.25 m gives a nondimensional impedance of  $Z = 2 - 0.35i$ . We continue with  $m = 0$  and the constant-then-linear boundary-layer profile for a fair comparison with [28,29]. For these parameters, a comparison of the various boundary conditions is shown in Fig. 8. Figure 8a shows the almost-propagating modes, and demonstrates that for



**Fig. 7** Comparison of the proposed boundary condition of Rienstra and Darau [Eq. (17)] with  $\tilde{\theta} = 0$  ([28], Eq. 25) or  $\tilde{\theta} = 1/3$  ([29], Eq. 4.4), the modified boundary condition proposed here [Eq. (16)], the unmodified Myers boundary condition [Eq. (5)], and the numerical solution, for a constant-then-linear velocity profile [Eq. (13)]. Parameters are as for [28] with a 0.2-m-diam cylindrical duct, giving  $M = 0.176$ ,  $\delta = 3.6 \times 10^{-4}$ , and  $Z = 2 + 3.9i\omega - i \cot(0.3\omega)$ , with  $\omega = 1$  and  $m = 0$  (giving  $Z = 2 + 0.667i$ ).



**Fig. 8** Comparison of the proposed boundary condition of Rienstra and Darau [Eq. (17)] with  $\tilde{\theta} = 0$  ([28], Eq. 25) or  $\tilde{\theta} = 1/3$  ([29], Eq. 4.4), the modified boundary condition proposed here [Eq. (16)], the unmodified Myers boundary condition [Eq. (5)], and the numerical solution, for a constant-then-linear velocity profile [Eq. (13)], for  $M = 0.5$ ,  $\delta = 10^{-2}$ ,  $\omega = 31$ ,  $Z = 2 + 0.016i\omega - i \cot(0.028\omega) = 2 - 0.35i$  and  $m = 0$ .

upstream-propagating modes (shown in more detail in Fig. 8c) the modified boundary condition given here again correctly predicts the attenuation rate of these modes (given by  $\text{Im}(k)$ ), while either of the predictions of Rienstra and Darau are less accurate. Figure 8d shows a similar plot for the low-radial-order downstream-propagating modes, which in this case shows the modified boundary condition proposed here to give superb accuracy. The surface mode shown in the lower-right of Fig. 8d is only correctly predicted by the boundary condition proposed here, with neither of Rienstra and Darau's boundary conditions predicting its existence. Interestingly, for the upstream-propagating cutoff modes (shown in Fig. 8b) with approximately  $50 \leq \text{Im}(k) \leq 90$ , none of the boundary conditions appears to accurately replicate the numerical solution. This situation looks to be similar to the situation where a surface mode is in close proximity to the cutoff acoustic modes, and indeed this may well be the case here. This is supported by the observation that this behavior is highly dependent on the value of the impedance used.

The preceding results are generally indicative of results obtained for a variety of impedances; for space reasons, not all results calculated are reproduced here. It is emphasized that the boundary condition proposed here [Eq. (9)] is capable of modelling three-dimensional perturbations with nonzero  $m$ , boundary-layer profiles other than constant-then-linear, and nonuniform boundary-layer densities. Since the boundary conditions of Rienstra and Darau [28,29] were derived assuming two-dimensional perturbations to a constant-then-linear uniform-density boundary layer, for nonzero  $m$  or for nonlinear boundary-layer profiles the boundary condition proposed here is expected to give even better agreement with numerical solutions than the boundary conditions of [28,29], although this is not investigated further here.

## VII. Conclusions

The main conclusion of this paper is that the new modified Myers boundary condition, derived asymptotically for thin boundary layers and given in Eq. (9) [with  $\delta I_0$  and  $\delta I_1$ , given in Eq. (8)], solves the ill-posedness problem associated with the standard Myers boundary condition. It can therefore be expected to alleviate the numerical instabilities associated with simulations using the Myers boundary condition in the time domain (applied, e.g., as suggested in Sec. III.C), as well as allowing a rigorous stability analysis (which is helpful for numerical simulations in the frequency domain).

The stability analyses conducted on the examples given here indicate that these examples are absolutely unstable for sufficiently thin boundary layers, and are convectively unstable otherwise. The only unstable mode found is the one predicted as being a hydrodynamic instability by Rienstra [33].

The new terms in the boundary layer are encapsulated within the  $\delta I_0$  and  $\delta I_1$  terms, which are integrals over the boundary layer and are valid for arbitrary boundary-layer profiles. The  $\delta I_0$  term may be calculated knowing only the mass, momentum, and kinetic energy thicknesses of the boundary layer. These three terms can be physically interpreted as a change in boundary mass, change in convected boundary speed, and a tension along the boundary. The  $\delta I_1$  term is more difficult to classify, but expressions are given in Sec. III.B in the high- and low-frequency limits, and an exact expression is given for  $\delta I_1$  for a constant-then-linear boundary-layer profile. It is the  $\delta I_1$  term that is responsible for regularizing the Myers boundary condition, though it should be stressed that both terms are important in order that the new boundary condition be asymptotically accurate. The  $\delta I_1$  term is also strongly linked to the presence of a critical layer at  $r = r^*$ , for which  $\omega - U(r^*)k = 0$ . The case  $r^* \in [0, 1]$  is explicitly excluded from our analysis, leading to a branch cut in the  $k$ -plane (for  $\omega$  considered fixed). For certain profiles, this branch cut may be removable, as is the case for a constant density and constant-then-linear boundary-layer profile given in Eq. (13), while for other profiles it may not be removable, possibly leading to algebraic growth [35]. The true nature and significance of the critical layer remains uncertain, and appears to depend strongly on the boundary-layer profile (see, e.g., [36,37]).

The asymptotics have been derived under the assumption that  $\mathcal{O}(\delta^2)$  quantities may be neglected in the boundary condition; this is valid provided  $k$ ,  $m$  and  $\omega$  remain  $\mathcal{O}(1)$ , or indeed provided they remain  $o(1/\delta)$ . For example, once  $k$  becomes sufficiently large that a wavelength becomes comparable to the boundary-layer thickness, the asymptotics described in this paper can be expected to be inaccurate, although the modified boundary condition will remain well posed. In light of this, it is interesting to note that the proposed boundary condition appears to remain accurate for the majority of modes for the parameters  $\omega = 31$  and  $\delta = 0.01$ , as used in Fig. 8, where it might have been argued a priori that  $\omega = \mathcal{O}(1/\delta)$ . It should also be noted that the unmodified Myers boundary condition was derived under the assumption that  $\mathcal{O}(\delta)$  quantities may be neglected, and therefore the range of validity of the modified Myers boundary condition proposed here is greater than that of the unmodified Myers boundary condition.

While the derivation here has been for a straight cylindrical duct, exactly the same analysis is possible for a rectangular duct, leading to exactly the same modified boundary condition (a brief note on the mathematical derivation of this is given in the Appendix). However, we have excluded curved ducts with nonparallel flows, for which the final term in the Myers boundary condition [Eq. (1)] is nonzero. Whether this term can be incorporated into a similar modified boundary condition, or indeed whether or not this term is correct for a permeable lining (see, e.g., [11]), is not considered here.

As well as regularizing the Myers boundary condition, the new boundary condition may be used to gain insight into the effect of a boundary layer on the modal wave numbers. The new boundary condition has been shown to be at least as accurate as the Myers boundary condition in all cases considered, and significantly more accurate in the majority of cases, as shown in Figs. 2, 4, 7, and 8. Moreover, when compared with the boundary conditions of Rienstra and Darau [28,29], the cuton acoustic modes are found more accurately with the boundary condition presented here, as seen in Figs. 7 and 8. The boundary condition of Rienstra and Darau [29] for  $\tilde{\theta} = 1/3$  is shown to be more accurate for one surface mode shown in Fig. 7b; however, the  $\tilde{\theta} = 1/3$  boundary condition fails to predict the existence of a different type of surface mode shown in Fig. 8d. The modified boundary condition presented here also has the advantage of being usable for three-dimensional perturbations for which  $m \neq 0$ , for nonlinear boundary-layer velocity profiles, for nonconstant boundary-layer temperature/density profiles, and for cylindrical as well as rectangular ducts.

For all examples considered here,  $R(r) \equiv 1$ , and so  $\delta_{\text{mass}} = 0$ . However, cases when  $\delta_{\text{mass}} \neq 0$  are both realizable in practice and potentially interesting mathematically, especially for  $\omega \gg 1$ , since  $\delta I_0$  includes an  $\omega^2 \delta_{\text{mass}}$  term. Such situations will be investigated further in future.

## Appendix: Derivation of the Boundary-Layer Asymptotics

In this Appendix, we derive the acoustic field within a thin boundary layer of width  $\delta$  correct to  $\mathcal{O}(\delta^2)$ . For those not interested in the derivation, a summary of the major results derived here is given in the final paragraph of this Appendix.

Our governing equation is the cylindrical form of the Pridmore-Brown equation [Eq. (A1)] for the acoustic pressure in a sheared flow:

$$\tilde{p}'' + \left( \frac{1}{r} + \frac{2kU'}{\omega - Uk} - \frac{R'}{R} \right) \tilde{p}' + \left( R(\omega - Uk)^2 - k^2 - \frac{m^2}{r^2} \right) \tilde{p} = 0 \quad (\text{A1})$$

with the radial velocity given by  $\tilde{v} = i\tilde{p}'/(R(\omega - Uk))$ . Here,  $U(r)$  is the mean-flow velocity and  $R(r)$  is the mean-flow density, nondimensionalized so that  $U(0) = M$ , the Mach number, and  $R(0) = 1$ . We consider a thin boundary layer about  $r = 1$  of typical width  $\delta$ , outside which the mean-flow is uniform, so that  $U(r) = M$  and  $R(r) = 1$  for  $r$  outside the boundary layer.



If the boundary layer did not exist, the solution to Eq. (A1) would be  $\tilde{p}(r) = \tilde{p}_o(r)$ :

$$\tilde{p}_o(r) = EJ_m(\alpha r) \quad \alpha^2 = (\omega - Mk)^2 - k^2$$

Expanding this for  $r = 1 - \delta y$ , and using Bessel's equation to write  $J_m''$  in terms of  $J_m$  and  $J_m'$ , gives

$$\begin{aligned} \tilde{p}_o(1 - \delta y) &= EJ_m(\alpha) - \delta y \alpha EJ_m'(\alpha) - \frac{1}{2} \delta^2 y^2 E[\alpha J_m''(\alpha)] \\ &+ (\alpha^2 - m^2) J_m(\alpha) + \mathcal{O}(\delta^3) \end{aligned} \quad (\text{A2})$$

This is our outer expansion, that the inner expansion within the boundary layer must match with.

To consider the boundary layer, it is helpful to first rearrange the governing equation to give

$$\left( \frac{r \tilde{p}'}{(\omega - Uk)^2 R} \right)' + \left( r - \frac{k^2 r + m^2/r}{(\omega - Uk)^2 R} \right) \tilde{p} = 0 \quad (\text{A3})$$

Substituting  $r = 1 - \delta y$  into Eq. (A3) and using a subscript  $y$  to denote  $d/dy$  gives

$$\begin{aligned} \left( \frac{\tilde{p}_y}{(\omega - Uk)^2 R} \right)_y &= \delta \left( \frac{y \tilde{p}_y}{(\omega - Uk)^2 R} \right)_y \\ &- \delta^2 \left( 1 - \frac{k^2 + m^2}{(\omega - Uk)^2 R} \right) \tilde{p} + \mathcal{O}(\delta^3) \end{aligned}$$

In a slight but obvious abuse of notation,  $U(y)$  is used to represent  $U(r)$  with  $r = 1 - \delta y$ , and similarly for  $R$ . We pose the solution  $\tilde{p} = \tilde{p}_0 + \delta \tilde{p}_1 + \delta^2 \tilde{p}_2 + \mathcal{O}(\delta^3)$ . To leading order,

$$\tilde{p}_0 = A_0 + B_0 y - B_0 \int_0^y 1 - \frac{(\omega - U(y')k)^2 R(y')}{(\omega - Mk)^2} dy'$$

Matching with the outer solution [Eq. (A2)] for large  $y$  at leading order gives  $A_0 = EJ_m(\alpha)$  and  $B_0 = 0$ . Since  $B_0 = 0$ , at first order we similarly find

$$\tilde{p}_1 = A_1 + B_1 y - B_1 \int_0^y 1 - \frac{(\omega - U(y')k)^2 R(y')}{(\omega - Mk)^2} dy'$$

while matching with the outer solution [Eq. (A2)] for large  $y$  gives  $B_1 = -\alpha EJ_m'(\alpha)$  and  $A_1 = B_1 I_0$ , with

$$I_0 = \int_0^\infty 1 - \frac{(\omega - U(y)k)^2 R(y)}{(\omega - Mk)^2} dy$$

At second order we find

$$\begin{aligned} \tilde{p}_2 &= A_2 + B_2 \int_0^y \frac{(\omega - U(y')k)^2 R(y')}{(\omega - Mk)^2} dy' \\ &+ B_1 \int_0^y y' \frac{(\omega - U(y')k)^2 R(y')}{(\omega - Mk)^2} dy' \\ &- A_0 \int_0^y (\omega - U(y')k)^2 R(y') \\ &\times \int_0^{y'} 1 - \frac{k^2 + m^2}{(\omega - U(y'')k)^2 R(y'')} dy'' dy' \end{aligned}$$

Rewriting this in terms of bounded integrals to aid matching with the outer solution gives

$$\begin{aligned} \tilde{p}_2 &= A_2 + B_2 y - B_2 \int_0^y 1 - \frac{(\omega - U(y')k)^2 R(y')}{(\omega - Mk)^2} dy' \\ &+ B_1 \int_0^y y' \left( \frac{(\omega - U(y')k)^2 R(y')}{(\omega - Mk)^2} - 1 \right) dy' + \frac{1}{2} B_1 y^2 \\ &- A_0 \int_0^y \left( \frac{(\omega - U(y')k)^2 R(y')}{(\omega - Mk)^2} - 1 \right) \int_0^{y'} (\omega - Mk)^2 - (k^2 + m^2) \\ &\times \frac{(\omega - Mk)^2}{(\omega - U(y'')k)^2 R(y'')} dy'' dy' - A_0 (k^2 + m^2) \\ &\times \int_0^y \left( \int_0^{y'} 1 - \frac{(\omega - Mk)^2}{(\omega - U(y'')k)^2 R(y'')} dy'' - I_1 \right) dy' \\ &- A_0 \frac{1}{2} ((\omega - Mk)^2 - k^2 - m^2) y^2 - A_0 I_1 (k^2 + m^2) y \end{aligned}$$

where the integrals are bounded provided we define

$$I_1 = \int_0^\infty 1 - \frac{(\omega - Mk)^2}{(\omega - U(y)k)^2 R(y)} dy$$

We now match this to the outer solution [Eq. (A2)] for large  $y$  by equating powers of  $y$ . The  $y^2$  terms match identically using our preceding definitions of  $A_0$  and  $B_1$ . Matching the  $y$  terms gives  $B_2 = A_0 I_1 (k^2 + m^2)$ , while matching the constant terms gives

$$\begin{aligned} A_2 &= B_2 I_0 + B_1 \int_0^\infty y \left( 1 - \frac{(\omega - U(y)k)^2 R(y)}{(\omega - Mk)^2} \right) dy \\ &+ A_0 \int_0^\infty \left( \frac{(\omega - U(y)k)^2 R(y)}{(\omega - Mk)^2} - 1 \right) \int_0^y (\omega - Mk)^2 \\ &- (k^2 + m^2) \frac{(\omega - Mk)^2}{(\omega - U(y')k)^2 R(y')} dy' dy + A_0 (k^2 + m^2) \\ &\times \int_0^\infty \left( \int_0^y 1 - \frac{(\omega - Mk)^2}{(\omega - U(y'')k)^2 R(y'')} dy'' - I_1 \right) dy \end{aligned}$$

Fortunately, for the modified Myers boundary condition we do not need to evaluate  $A_2$ !

## I. Summary

Now that we have the inner solution through the boundary layer, we may calculate  $\tilde{p}$  and  $\tilde{v}$  at the wall  $r = 1$ ,  $y = 0$ . Using  $\tilde{v} = i \tilde{p}' / (R(\omega - Uk))$ , we find that

$$\begin{aligned} \tilde{p}(1) &= EJ_m(\alpha) - \alpha EJ_m'(\alpha) \delta I_0 + \mathcal{O}(\delta^2) \\ \tilde{v}(1) &= \frac{i(\omega - U(1)k)}{(\omega - Mk)^2} [\alpha EJ_m'(\alpha) \\ &- (k^2 + m^2) \delta I_1 EJ_m(\alpha) + \mathcal{O}(\delta^2)] \end{aligned}$$

where we may rewrite  $\delta I_0$  and  $\delta I_1$  in terms of integrals over  $r$  as

$$\begin{aligned} \delta I_0 &= \int_0^1 1 - \frac{(\omega - U(r)k)^2 R(r)}{(\omega - Mk)^2} dr \\ \delta I_1 &= \int_0^1 1 - \frac{(\omega - Mk)^2}{(\omega - U(r)k)^2 R(r)} dr \end{aligned}$$

To first order, we may form the composite solution for the pressure:

$$\tilde{p}_c(r) = J_m(\alpha r) - \alpha J'_m(\alpha) \int_0^r 1 - \frac{(\omega - U(r)k)^2 R(r)}{(\omega - Mk)^2} dr. + \mathcal{O}(\delta^2)$$

## II. Derivation for a Rectangular Duct

For a rectangular duct, we take a solutions of the form  $\tilde{p} = \tilde{p}(r) \exp\{i\omega t - ikx - imz\}$ , where  $m$  is no longer restricted to taking integer values,  $z$  is the coordinate perpendicular to the flow in the plane of the liner, and  $r$  is the coordinate normal to the liner. The equivalent of the Pridmore-Brown Eq. (A1) in this case is

$$\tilde{p}'' + \left( \frac{2kU'}{\omega - Uk} - \frac{R'}{R} \right) \tilde{p}' + (R(\omega - Uk)^2 - k^2 - m^2) \tilde{p} = 0$$

Using exactly the same preceding procedure leads to exactly the same result as derived previously, although the algebra is slightly different since the  $1/r$  geometric term is not present in this case.

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